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The Disposition of an Ionized Gas in a Gravitational Field.

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Summary. — Calculations are given of the density distributions of electrons and ions in thermodynamic equilibrium in the presence of a uniform gravational field. Charge separation occurs and the resulting electrostatic potential has been calculated.

1. – Introduction.

The subject to be discussed in this paper is that of an ionized gas in thermodynamic equilibrium in the presence of a uniform gravitational field. This study has been made mainly for didactic purposes but it may also have some connection with the acceleration of plasmas, a subject which is attracting considerable interest.

In the presence of a gravitational field it is evident that the positive ions will tend to settle below the electrons since they are much heavier. A separation of the two species, however, produces an electrostatic field which opposes further charge separation. Thus the density distributions are determined in part by the gravitational field and in part by the electrostatic field. A magnetic field may or may not be present since the density distributions, in thermal equilibrium, are not affected by a magnetic field.

The field of acceleration g is taken to be in the negative x direction and the plasma is limited to the half-space $x > 0$ by a plane wall, so that the problem has plane symmetry. The distributions of potential, electron density and ion density are to be found for the boundary conditions $E(0) = 0$, $E(\infty) = 0$, where E is the electrostatic field. These conditions correspond to an uncharged wall at $x = 0$ and to the total amount of charge being zero.

In thermodynamic equilibrium, the densities of electrons and ions are given respectively by

(1) $$n_1 = n_{10} \exp[(-m_1 gx + eV)/kT] ,$$

and

(2) $$n_2 = n_{20} \exp[(-m_2 gx - eV)/kT] ,$$

where the indices 1 and 2 refer to electrons and ions, m denotes the mass, $E = -\mathrm{d}V/\mathrm{d}x$ and e is the absolute value of the elementary charge (for simplicity the ions are supposed singly charged). V is taken to be zero at the wall and so n_{10} and n_{20} are the electron and ion densities at the wall. Poisson's equation can be written down at once and has the following form:

(3) $$\frac{\mathrm{d}^2 V}{\mathrm{d}x^2} = 4\pi e \left\{ n_{10} \exp\left[\frac{(-m_1 gx + eV)}{kT}\right] - n_{20} \exp\left[\frac{(-m_2 gx - eV)}{kT}\right]\right\} .$$

A simple solution of this equation is the following

(4) $$V = -E_0 x ,$$

where

$$E_0 = (m_2 - m_1) g/2e ,$$

and

(5) $$n_1 = n_2 = n_0 \exp[-(m_2 + m_1) gx/2kT] .$$

This solution does not satisfy the boundary conditions but it corresponds instead to the case where the electric field is produced by charges at the boundaries. The gas is electrically neutral and a unique scale height exists which is given by $H = 2kT/[(m_1 + m_2) g]$. This solution can be referred to as the « plasma solution » since it corresponds to the case where the electron and ion densities are equal. It is shown below that this type of solution represents a first approximation to the true solution when $H \gg \lambda_D$, where λ_D is the Debye distance. The complete solution is also discussed in detail and is shown to be « universal » in the limit where $H \gg \lambda_D$.

2. – The divergence from the « plasma solution ».

It is convenient to introduce a function $\Phi(x)$ such that $(kT/e)\Phi$ is the difference between the true potential and that corresponding to a « plasma solution ». Thus the potential can be written in the form

(6) $$V = -E_0(x - x_0) + (kT/e)\Phi ,$$

where $x_0 = kT \ln (n_{20}/n_{10})/[(m_2 - m_1)g]$, so that

$$n_{10} \exp[-m_1 g x_0/kT] = n_{20} \exp[-m_2 g x_0/kT] = n_0 \quad \text{(say)} .$$

Poisson's equation now takes the following form

(7) $$\mathrm{d}^2\Phi/\mathrm{d}r^2 = a^2 \exp[-r] \sinh \Phi ,$$

where $r = x/H$, $\ln a^2 = (x_0/H + p)$ and the boundary conditions are

(8) $$\Phi(0) = \mu(p - \ln a^2) , \qquad (\mathrm{d}\Phi/\mathrm{d}r)_{r=0} = \mu ,$$

and

(9) $$(\mathrm{d}\Phi/\mathrm{d}r)_{r=\infty} = \mu ,$$

where $p = 2 \ln (H/\lambda_D)$, $\lambda_D^2 = kT/(8\pi n_0 e^2)$ and $\mu = (m_2 - m_1)/(m_2 + m_1)$. If Φ is a well-behaved function at infinity, then

(10) $$(\mathrm{d}^2\Phi/\mathrm{d}r^2)_{r=\infty} = 0 .$$

Physically this means that there is no space charge at infinity.

It should be noted that a is an unknown eigenvalue whereas p and μ are parameters whose values are given. For each pair of values of p and μ, there will be a solution Φ and a corresponding value of a. When the latter have been determined, x_0, V, n_1, n_2 can be found from the following

(11) $$x_0 = -H\Phi(0)/\mu ,$$

(12) $$V = (kT/e)\{-\mu[r + \Phi(0)/\mu] + \Phi\} ,$$

(13) $$n_1 = n_0 \exp\left[-(r + \Phi(0)/\mu) + \Phi\right] ,$$

(14) $$n_2 = n_0 \exp\left[-(r + \Phi(0)/\mu) - \Phi\right] ,$$

which are a consequence of the definitions. From these equations it follows that $n_2 = n_1 \exp[-2\Phi]$.

It can be verified that the solution is unique.

3. – Solution for small values of p.

For small values of p the solution can be obtained by means of the following procedure. Equation (7) can be integrated numerically, starting from $r = 0$ and with a value of a which is simply a guess. The integration can be continued until large values of r are reached so that the corresponding value of $\mathrm{d}\Phi/\mathrm{d}r$ may be compared with the required value $(\mathrm{d}\Phi/\mathrm{d}r) = \mu$. Hence by trial and error it is possible to determine a and to calculate the solution to the problem.

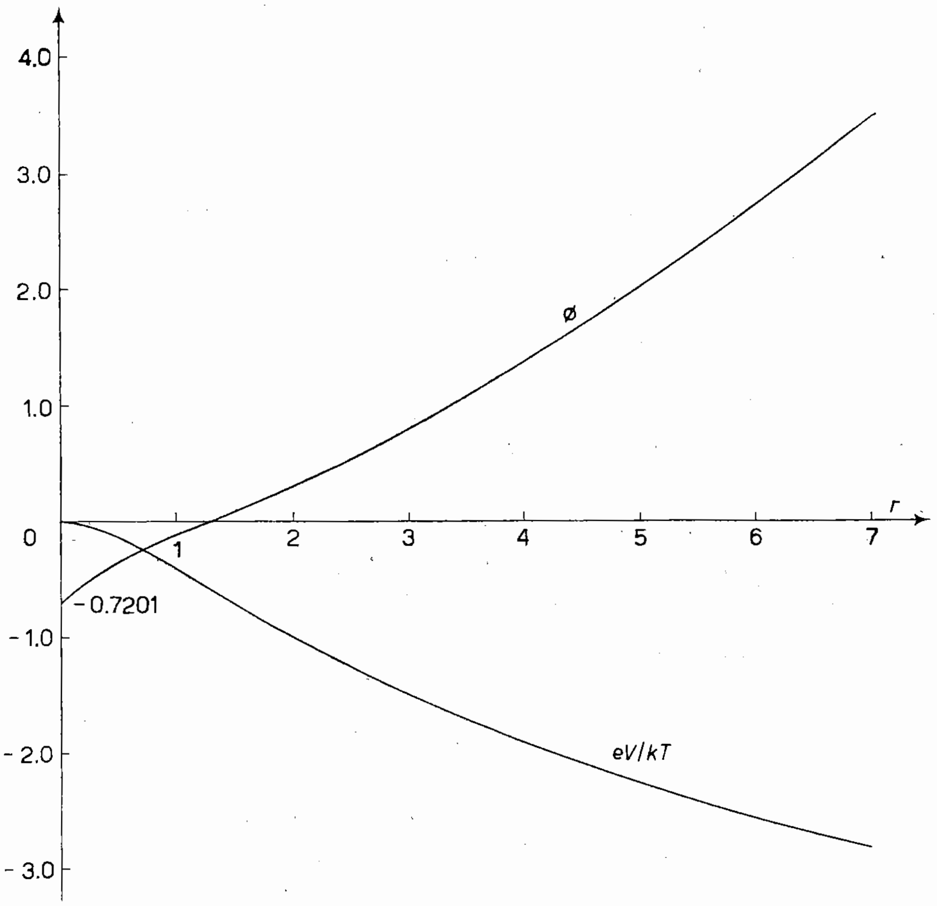

Fig. 1. – The function Φ and the total (normalized) potential (eV/kT) for the case where $H = \lambda_{\mathrm{D}}$ and $\mu = 1$.

As an example, this procedure has been followed for the case $p = 0$, $\mu = 1$, the integration being carried on up to $r \simeq 100$, and the results are given in Fig. 1 and 2. When p is small there is no extended «plasma» region where $n_1 \simeq n_2$. For larger values of p, however, such a region occurs and the nec-

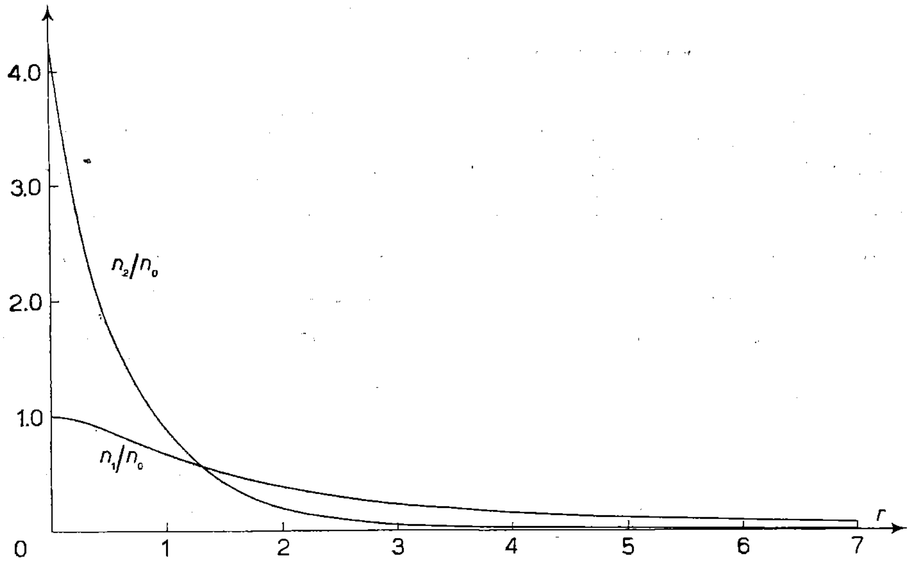

Fig. 2. – The electron and ion density distributions for the case where $H=\lambda_D$ and $\mu=1$.

essary precision and length of the numerical integrations rapidly become prohibitive. Therefore in the following paragraphs the asymptotic solution for $p\to\infty$ will be discussed.

4. – Solution for $p=2\ln(H/\lambda_D)\gg 1$.

4·1. *Asymptotic solutions.* – From the form of eq. (7) and the boundary conditions (8) and (9) it can be seen that (whatever the value of p) $\Phi(0)<0$ and $0<\mathrm{d}\Phi/\mathrm{d}r\leqslant\mu$.

It will be assumed that

$$|\Phi(0)|\ll 1\,, \tag{15}$$

and hence, using eq. (8), $a^2\simeq\exp[p]$, *i.e.* $a^2\gg 1$. Later it will be shown that this assumption is consistent with the solution. From condition (15) it follows that, starting at $r=0$ and up to some value of r, eq. (7) can be substituted by the following equation:

$$\mathrm{d}^2\Phi/\mathrm{d}r^2=a^2e^{-r}\Phi\,. \tag{16}$$

This equation can be integrated and the solution which satisfies the boundary conditions (8) is the following:

$$\Phi=c_1I_0(z)+c_2K_0(z)\,, \tag{17}$$

and

$$\mathrm{d}\Phi/\mathrm{d}r = -(c_1/2)zI_1(z) + (c_2/2)zK_1(z)\,, \tag{18}$$

where $I_0(z)$ and $K_0(z)$ are the modified Bessel functions of the first and second kind and order zero, and $I_1(z)$ and $K_1(z)$ are those of order one,

$$z = 2a\exp[-r/2]\,, \tag{19}$$

and c_1, c_2 are constants given by

$$c_1 = -2[\mu k_0 - \Phi(0)\,ak_1]\,,$$

$$c_2 = 2[\mu i_0 + \Phi(0)\,ai_1]\,,$$

where

$$\left.\begin{aligned} i_n &= I_n(2a)\,,\\ k_n &= K_n(2a)\,,\end{aligned}\right\} \qquad (n = 0, 1).$$

Use has been made of the property that

$$I_0(z)K_1(z) + I_1(z)K_0(z) = 1/z\,.$$

In the limit $a \gg 1$

$$i_0 = i_1 = (4\pi)^{-\frac{1}{2}}a^{-\frac{1}{2}}\exp[2a]\,,$$

$$k_0 = k_1 = (\pi/4)^{\frac{1}{2}}a^{-\frac{1}{2}}\exp[-2a]\,,$$

and so

$$c_1 = -(\mu - \Phi(0)\,a)\,\pi^{\frac{1}{2}}a^{-\frac{1}{2}}\exp[-2a]\,, \tag{20}$$

$$c_2 = (\mu + \Phi(0)\,a)\pi^{-\frac{1}{2}}a^{-\frac{1}{2}}\exp[2a]\,. \tag{21}$$

In the limit $a \to \infty$, as c_2 must remain finite, it is clear that $[\mu + \Phi(0)a] \to 0$. Therefore

$$\Phi(0) = -\mu/a\,, \tag{22}$$

and using eq. (8), the following relations are obtained (in the limit $a \gg 1$):

$$a = \exp[p/2] = (H/\lambda_{\mathrm{D}})\,, \tag{23}$$

$$\Phi(0) = -\mu\lambda_{\mathrm{D}}/H\,. \tag{24}$$

From eqs. (22) and (20) it follows that

$$c_1 = -2\mu\pi^{\frac{1}{2}}a^{-\frac{1}{2}}\exp[-2a]\,. \tag{25}$$

The value of c_2 is as yet indeterminate and will be determined by the use of condition (9). It can be seen that eq. (24) justifies assumption (15).

From eqs. (11) and (24) it follows that $x_0 = \lambda_D$ and, using the asymptotic forms of I_0 and K_0 for large values of the argument, it can be seen that for $r < 1$

$$\Phi \simeq -(\mu\lambda_D/H)\exp[-x/\lambda_D]\,. \tag{26}$$

Eventually, *i.e.* for sufficiently large values of r, the solution of eq. (7) gives values of Φ which are much greater than unity. It is convenient, therefore to consider the approximate form of eq. (7) for $\Phi \gg 1$, *i.e.*

$$d^2\Phi/dr^2 = (a^2/2)\exp[\Phi - r]\,. \tag{27}$$

This equation can be integrated once and, using conditions (9), (10), the following first integral is obtained:

$$(1 - d\Phi/dr)^2 = a^2\exp[\Phi - r] + (1-\mu)^2\,. \tag{28}$$

A second integration gives

$$r - r_1 = \frac{1}{(1-\mu)}\ln\frac{\left[\sqrt{(1-\mu)^2 + a^2\exp[(\Phi - r)]} + (1-\mu)\right]}{\left[\sqrt{(1-\mu)^2 + a^2\exp[(\Phi - r)]} - (1-\mu)\right]}\,, \tag{29}$$

where r_1 is a constant of integration. It is convenient to consider the two regions, I and II, where $(1-\mu)^2$ is respectively much less or much greater than $a^2\exp[\Phi - r]$.

For $(1-\mu)^2 \ll a^2\exp[\Phi - r]$, eq. (29) reduces to the following

$$\text{I)}\qquad \Phi = r - 2\ln(r - r_1) + 2\ln 2 - 2\ln a\,, \tag{30}$$

and eq. (28) to

$$(1 - d\Phi/dr)^2 = a^2\exp[\Phi - r]\,. \tag{31}$$

For $(1-\mu)^2 \gg a^2\exp[\Phi - r]$, eq. (29) reduces to the following:

$$\text{II)}\qquad \Phi = \mu r + (1-\mu)r_1 - 2\ln a + 2\ln[2(1-\mu)]\,. \tag{32}$$

The transition between these two regions occurs where

$$(1-\mu)^2 = a^2 \exp[\Phi - r]\,,$$

and therefore, using eq. (29), at $r = r_2$, where

$$r_2 = r_1 + \frac{1}{(1-\mu)} \ln \frac{\sqrt{2}+1}{\sqrt{2}-1} = r_1 + \frac{0.763}{(1-\mu)}\,. \tag{33}$$

4·2. *Joining together of the asymptotic solutions.* – As long as $\Phi \ll 1$, the solution is given by eq. (17) where c_1 is given by eq. (25) and c_2 is to be determined by joining solution (17) to a solution of the form given by eq. (29) across a region of r where $\Phi \sim 1$. It is shown in the following that, when $(1-\mu)$ is sufficiently small, r_2 is so large that solution (17) can be joined to a solution of the form of eq. (30).

The following procedure may now be adopted: starting with a conjectured value of c_2, eq. (7) can be integrated numerically, starting from a value of z and hence r (z_0 and r_0, say) such that $\Phi(z_0) \ll 1$. The integration can be carried out until large values of Φ are obtained and the calculated $\Phi(r)$ can then be compared with the required dependence on r given by eq. (31). Thus, by trial and error, one can determine c_2 and the solution between the regions where $\Phi \ll 1$ and where $\Phi \gg 1$. This procedure is simplified by the consideration that, if one assumes, as will be confirmed by the results, that $\lim_{a\to\infty} c_2 \neq 0$, then, for $a \to \infty$, $\Phi(z_0) = c_2 K_0(z_0)$ and $(\mathrm{d}\Phi/\mathrm{d}r)_{z=z_0} = (c_2/2) z_0 K_1(z_0)$. Therefore eq. (7) is to be integrated with the following initial conditions:

$$r_0 = 2 \ln a + \ln 4 - 2 \ln z_0\,, \tag{34}$$

$$\Phi(r_0) = c_2 K_0(z_0)\,, \tag{35}$$

$$(\mathrm{d}\Phi/\mathrm{d}r)_{r=r_0} = (c_2/2) z_0 K_1(z_0)\,, \tag{36}$$

where z_0 is chosen so that $\Phi(z_0) \ll 1$. On introduction of a new variable $\varrho = (r - 2 \ln a)$, eq. (7) and conditions (34), (35), (36) become

$$\mathrm{d}^2\Phi/\mathrm{d}\varrho^2 = \exp[-\varrho] \sinh \Phi\,, \tag{37}$$

$$\varrho_0 = \ln 4 - 2 \ln z_0\,, \tag{38}$$

$$\Phi(\varrho_0) = c_2 K_0(z_0)\,, \tag{39}$$

$$(\mathrm{d}\Phi/\mathrm{d}\varrho)_{\varrho=\varrho_0} = (c_2/2) z_0 K_1(z_0)\,. \tag{40}$$

It is important to note that expressions (37), (38), (39), (40) do not contain p and that, when the specification (to be stated more precisely) that r_2 be large is met, then the integration also does not involve μ. In this case the values of c_2 and the numerical solution $\Phi(\varrho)$ are universal.

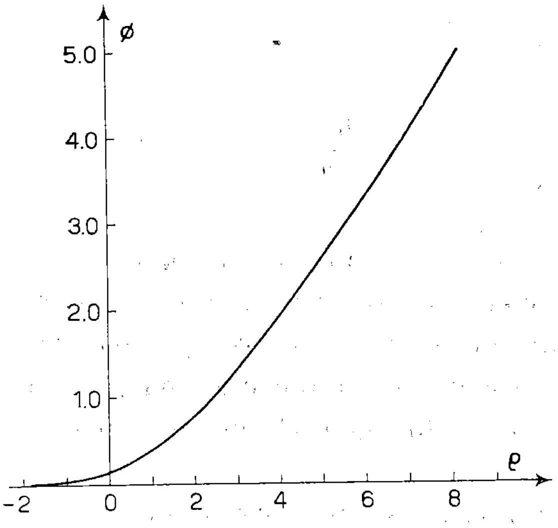

Fig. 3. – The function $\Phi(\varrho)$, where $\varrho = r - 2\ln(H/\lambda_D)$, for the case where $H \gg \lambda_D$. This curve does not depend on μ; such a dependence exists, however, at larger values of ϱ.

For the numerical calculation of $\Phi(\varrho)$, a value $z_0 = 4.9798$ was chosen so that $\varrho_0 = -1.8245$, $K_0(z_0) = 3.7731 \cdot 10^{-3}$ and $K_1(z_0) = 4.1360 \cdot 10^{-3}$. The integration was performed for a series of values of c_2. It was found that the best fit occurs for $1.2730 < c_2 < 1.2731$ and the functions obtained using these limiting values for c_2 differ by less than 0.03% for values of ϱ up to 35.04. Figure 3 shows the calculated curve $\Phi(\varrho)$.

It is now possible to use the final points of the numerical integration, in order to determine the value of r_1 in eq. (30), and hence the value of r_2 in eq. (33). From the numerical integration one obtains $\Phi = 29.21$ for $\varrho = 35.04$. Hence, using eq. (30),

$$r_1 = 2\ln a + \varrho - 2\exp[(\varrho - \Phi)/2] = 2\ln a - 1.86\,, \tag{41}$$

and, from eq. (33),

$$r_2 = 2\ln a - 1.86 + 1.763/(1-\mu)\,. \tag{42}$$

Having determined c_2, it can be shown, by use of the asymptotic forms of $I_0(z)$, $K_0(z)$ for $z \gg 1$ in eq. (17), that $\Phi = 0$ for $r = 2\ln 2$ (in the limit $a \to \infty$) and so $z = a$ at this point, *i.e.* $z \gg 1$.

It is now possible to specify more precisely what is meant by the requirement that r_2 be sufficiently large. For the existence of a region where eq. (30) holds, r_2 must be so great that $(r_2 - 2\ln a) \gg 1$, *i.e.* from eq. (42) $(1-\mu) \ll 1$. This condition is generally satisfied. Using eq. (41) in eqs. (30) and (32), respectively, the following are obtained:

$$\text{I)}\qquad \Phi = r - 2\ln a + 1.39 - 2\ln[r - 2\ln a + 1.86]\,, \tag{43}$$

$$\text{II)}\qquad \Phi = \mu(r - 2\ln a + 1.86) - 1.86 + 2\ln[2(1-\mu)]\,. \tag{44}$$

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4·3. *Calculation of physical quantities.* – Having determined Φ, eqs. (12), (13) and (14) can be used to determine V, n_1 and n_2. Figure 4 is a schematic diagram showing the variation of Φ and V with r and the regions of applicability of the various asymptotic equations.

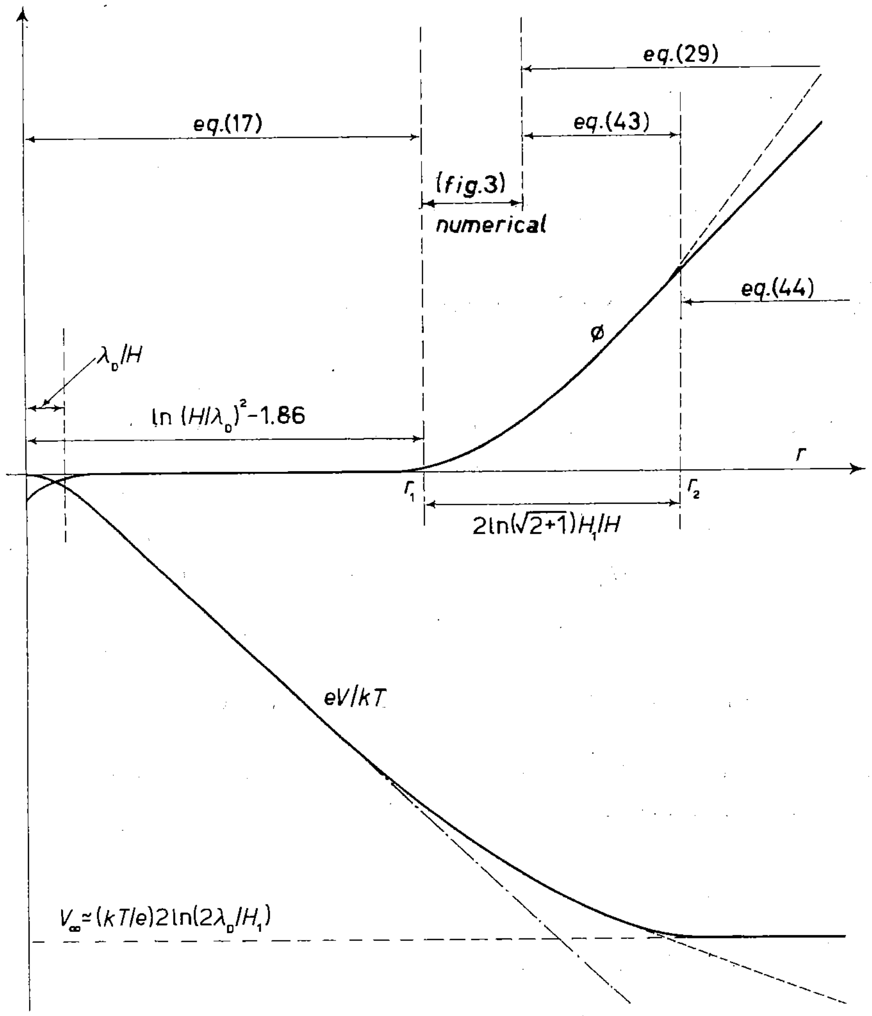

Fig. 4. – Schematic diagram (not to scale) showing the function Φ and the total potential V for the case where $H \gg \lambda_D$. The regions of applicability of the various asymptotic equations are shown and the region of numerical integration is indicated. The dashed curve indicates the solution for $\mu = 1$ (for $r > r_2$).

For $r \ll r_1$, Φ is practically zero so that

(45) $$V \simeq -(kT/e)\mu r = -E_0 x\,,$$

(46) $$n_1 \simeq n_0 \exp[-r]\,,$$

and

(47) $$n_2 \simeq n_0 \exp[-r] .$$

This is the simple « plasma solution » given in the introduction (eqs. (4) and (5)).

For $r_1 \ll r \ll r_2$,

(48) $$V = (kT/e)[(1-\mu)r - 2\ln a + 1.39 - 2\ln(r - 2\ln a + 1.86)] ,$$

(49) $$n_1 = n_0 \alpha (\lambda_D/H)^2 r_*^{-2} ,$$

and

(50) $$n_2 = n_0 \beta (\lambda_D/H)^2 r_*^2 \exp[-2r_*] ,$$

where $\alpha = \exp[1.39]$, $\beta = \exp[2.33]$ and $r_* = r - r_1$.

For $r_2 \ll r$,

(51) $$\begin{cases} V = (kT/e)\{(-2\ln a + 1.86)\mu - 1.86 + 2\ln[2(1-\mu)]\} , \\ \quad \simeq (kT/e)\, 2\ln(2\lambda_D/H_1) , \end{cases}$$

(52) $$n_1 \propto \exp[-x/H_1] ,$$

and

(53) $$n_2 \propto \exp[-x/H_2] ,$$

where $H_1 = kT/(m_1 g)$ and $H_2 = kT/(m_2 g)$ are the gravitational scale heights of electrons and ions respectively.

5. – Discussion.

The calculations reported above, for the case where $(H/\lambda_D)^2 \gg 1$, show that the plasma region, where the electron and ion densities are very nearly equal, extends up to a height of

$$H[\ln(H/\lambda_D)^2 - 1.86] \simeq H \ln(H/\lambda_D)^2 .$$

Except for within a thin layer $\sim \lambda_D$, which is associated with the required « surface charge », the plasma scale height $2kT/[(m_1 + m_2)g]$ governs the exponential decrease in plasma density with height; the electric field is given by $E_0 = (m_2 - m_1)g/2e$. The plasma density decreases until, at the height $H \ln(H/\lambda_D)^2$, the local Debye distance becomes comparable with the characteristic distance H, whereupon quasi-neutrality no longer obtains.

At greater distances the electron space-charge predominates. The electron cloud itself is divided into two regions. In the first region the gravitational force on the electrons is negligible and they are essentially in equilibrium with their own space-charge field. The relatively small number of ions present in this region have a spatial distribution determined by both gravity and the electric field.

At very great heights, *i.e.* above $H[\ln(H/\lambda_D)^2 - 1.86] + 2\ln(\sqrt{2}+1)H_1$, where H_1 is the electron scale height $kT/m_1 g$, the electric field becomes vanishingly small. Here both electrons and ions are distributed according to their individual scale heights. It is interesting to note that a finite potential difference exists between the top of the electron cloud and the base of the plasma.

* * *

Thanks are due to Dr.s M. Locci and T. Matitti for the numerical calculations in Sections **3** and **4**.2.

RIASSUNTO

Si presentano i calcoli delle distribuzioni di elettroni e ioni in equilibrio termodinamico, in presenza di un campo gravitazionale uniforme. Esiste separazione di cariche ed è stato calcolato il potenziale elettrostatico che ne risulta.